Path Covers of Temporal Graphs: When is Dilworth dynamic?

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School of Computing, University of Leeds

Algorithmic Aspects of Temporal Graphs VII
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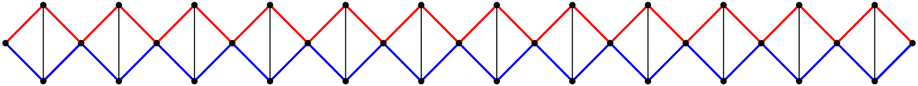


Joint work with

Figure: (from left) Dr. Antoine Dailly, Dr. Florent Foucaud, Prof. Ralf Klasing

- LIMOS, University of Clermont-Ferrand, France.
- School of Computing, University of Leeds, UK.
- LaBRI, University of Bordeaux.

Thanks to Antoine for the slides !

- Path Cover: A set of Paths that Cover all Vertices of a (di)graph.

Path Cover Problem

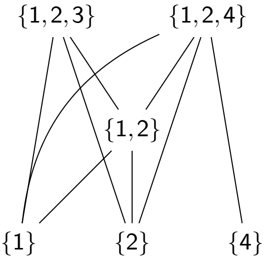
Input: A (di)graph

Output: A Path Cover of min. Size.

→ Hard in general.

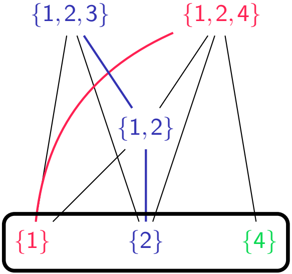
→ Polynomial time Solvable in DAGs.

Introduction: Dilworth's theorem



Theorem [Dilworth, 1950]

The minimum size of a chain partition of a finite poset is equal to the maximum size of an antichain of this poset.

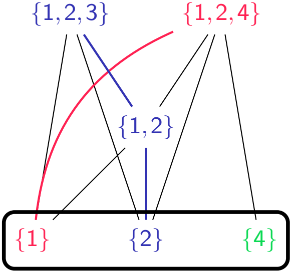


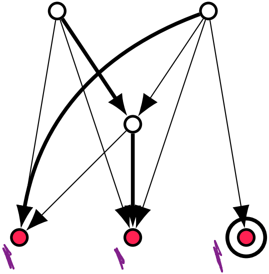
Introduction: Dilworth's theorem

Theorem [Dilworth, 1950]

The minimum size of a path partition of a transitive DAG is equal to the maximum size of an antichain of this DAG.

Restated for graphs...



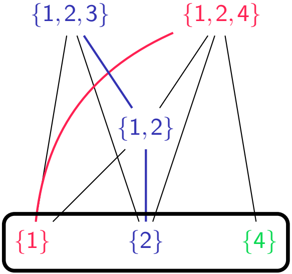


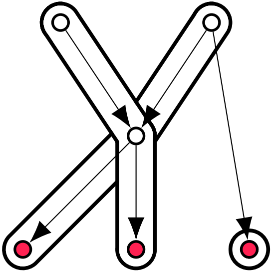
Introduction: Dilworth's theorem

Theorem [Dilworth, 1950]

The minimum size of a path cover of a DAG is equal to the maximum size of an antichain of this DAG.

Restated for graphs...
... and covers.





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Restated for graphs...
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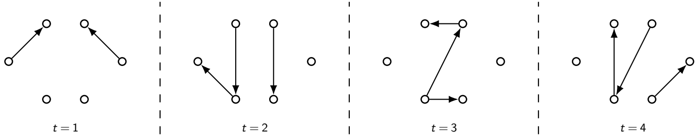
Algorithms:

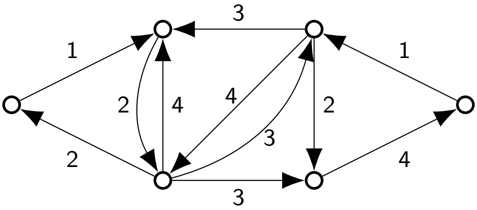
- Algorithmic proof (polynomial time) [Fulkerson, 1956]

A temporal analogue of Dilworth's theorem?

$$\mathscr{D} = (V, A_1, A_2, \ldots, A_k)$$

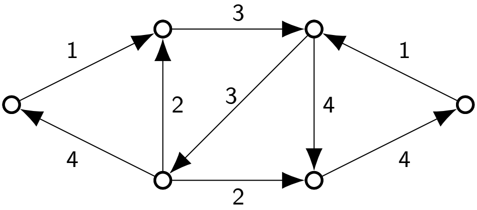


$$\mathscr{D} = (V, A, \lambda)$$



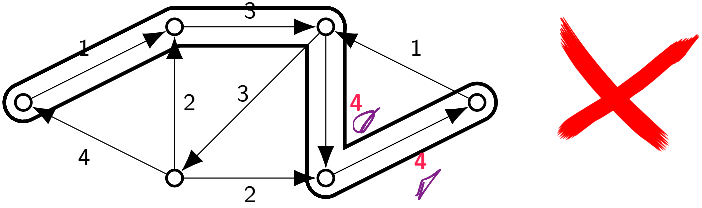
Many results and applications in distributed algorithms, dynamic networks (transportation, social, biological...), interest in the graph algorithms community.

- A temporal DAG (resp. tree...) is a temporal (di)graph whose underlying (di)graph is a DAG (resp. tree...).



A few definitions for this talk

- A temporal DAG (resp. tree...) is a temporal (di)graph whose underlying (di)graph is a DAG (resp. tree...).
- (Directed) temporal path : strictly increasing time labels.



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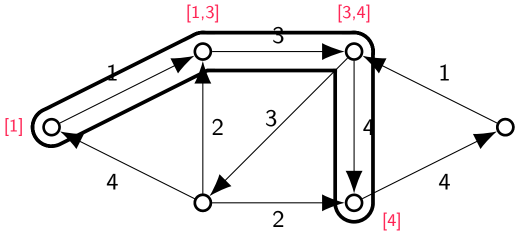

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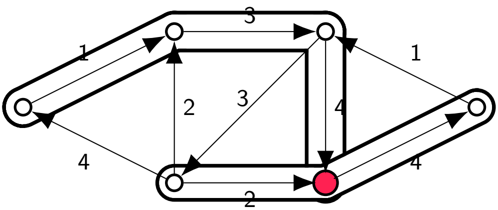

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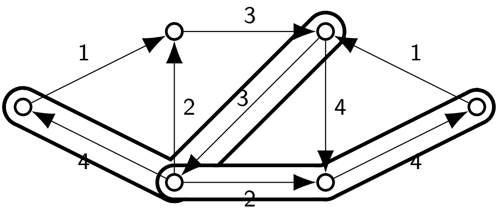

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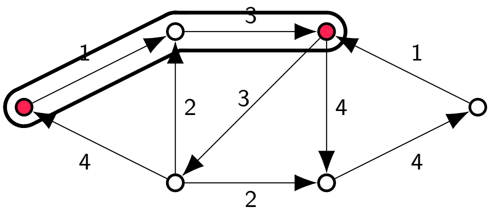

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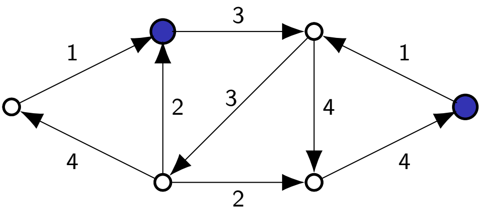
- A temporal antichain is a set of vertices who are pairwise not temporally connected.

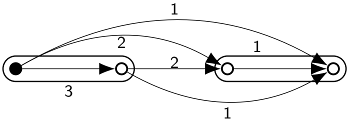


A temporal Dilworth's theorem?

Temporal Dilworth property?

In a temporal DAG, the minimum size of a temporal path partition/cover is equal to the maximum size of a temporal antichain.

Not all temporal DAGs have the Temporal Dilworth Property.



A temporal Dilworth's theorem?

Temporal Dilworth property?

In a temporal DAG, the minimum size of a temporal path partition/cover is equal to the maximum size of a temporal antichain.

Two problems:

Temporal Path Cover (TPC)

Temporal Path Partition/Temporally Disjoint Path Cover (TD-PC)

Two questions:

Which temporal DAGs have the Dilworth property?

$\Rightarrow$ **Combinatorial** aspect

What is the complexity of those problems?

$\Rightarrow$ **Algorithmic** aspect

Our results

⟨Temporal Path Cover⟩ ⟨Temporally - Disj. Path Cover⟩

Temporal class	**TPC**	**TD-PC**
Oriented paths	$\mathcal{O}(\ell n)$	$\mathcal{O}(\ell n)$
Rooted trees	$\mathcal{O}(\ell n^2)$	$\mathcal{O}(\ell n^2)$
Oriented trees	$\mathcal{O}(\ell n^2 + n^3)$	NP-hard
DAGs*	NP-hard	NP-hard
Digraphs	XP (tw and $t_{\max}$) $n^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))}$	FPT (tw and $t_{\max}$) $2^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))}\, n$

* planar, subcubic, bipartite, girth 10, $\ell = 1$, $t_{\max} = 2$

$$n = \text{number of vertices}$$
$$\ell = \text{number of (unsorted) time labels per arc}$$
$$t_{\max} = \text{total number of time-steps}$$

Classes with polynomial-time algorithm also have the Dilworth property.

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Not difficult

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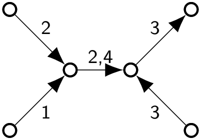
Classes with polynomial-time algorithm also have the Dilworth property.

Theorem [CDFK, 2024+]

Temporal oriented trees have the Dilworth property for TPC, and we can solve TPC in time $\mathcal{O}(\ell n^2 + n^3)$.

$\mathscr{T} =$ 

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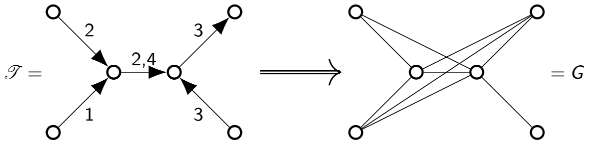
Algorithm

- Construct an auxiliary connectivity graph: two vertices are adjacent $\Leftrightarrow$ they are temporally connected in the tree



Min
- clique Cover in the Connectivity graph
- (Min clique Cover) $\Leftrightarrow$ (Min Path Cover) of the tree.

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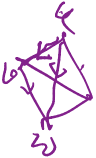

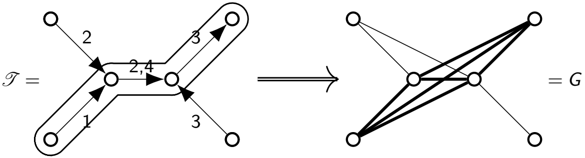

Lemma

Clique in G $\Leftrightarrow$ Temporal Path in $\mathcal{T}$.

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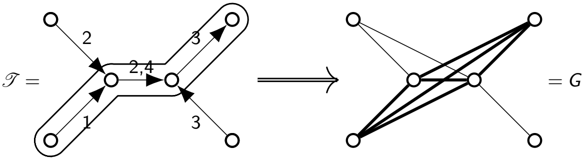

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Path Cover of temporal oriented trees (2) Holes

Lemma

There are no holes of length greater than four in the connectivity graph.

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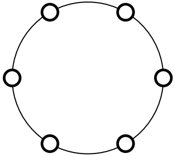
Claim

The vertices of the hole are leaves of a connected subtree T'.



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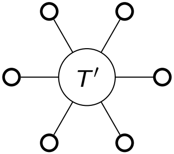

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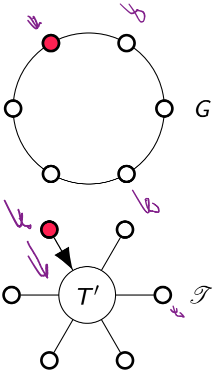
Alternating between in-arcs and out-arcs from and to T'.



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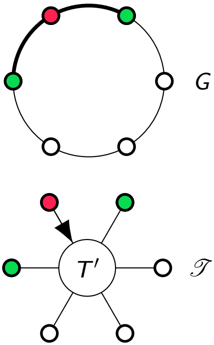

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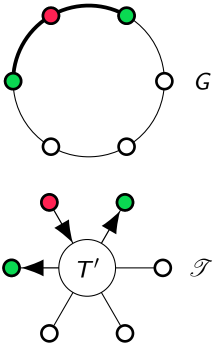

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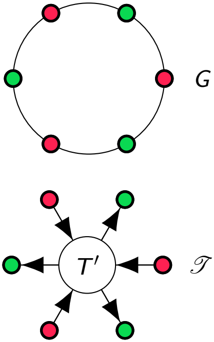

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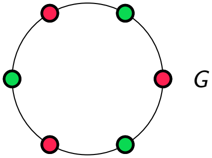

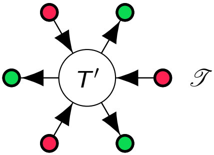

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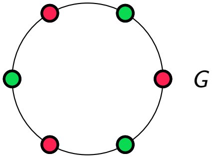
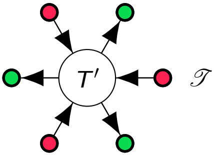
Claim

No even holes of length greater than four either (using Helly property and vertex-intersection of temporal paths).





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There are no antiholes in the connectivity graph.

Complement of holes ≥ 5

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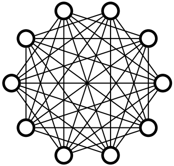
G



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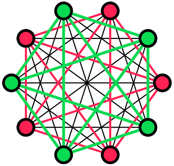

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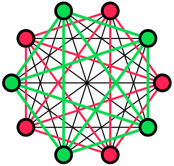
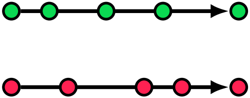
G  $\implies \mathscr{T}$ 

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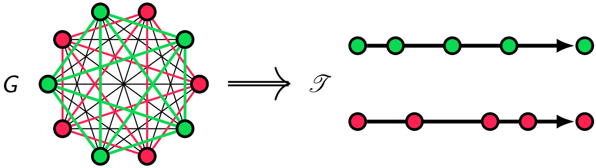
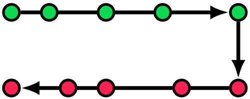


Case 1



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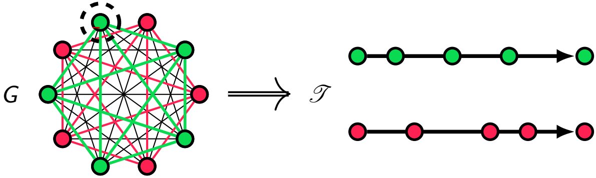

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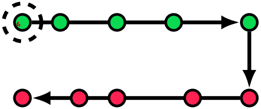

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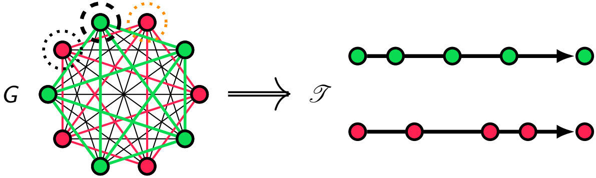

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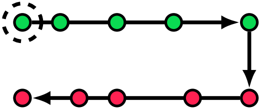

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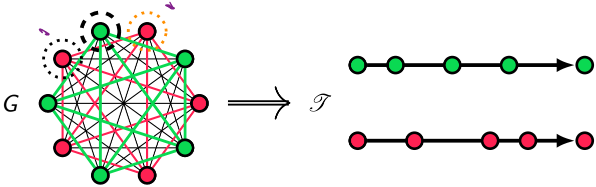

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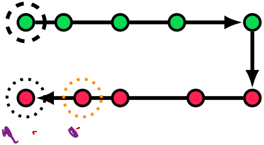

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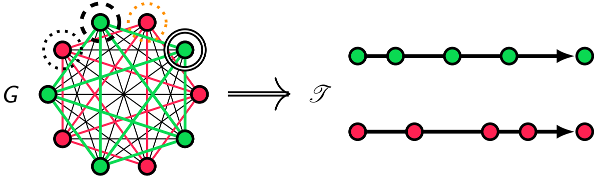

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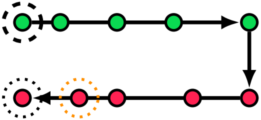

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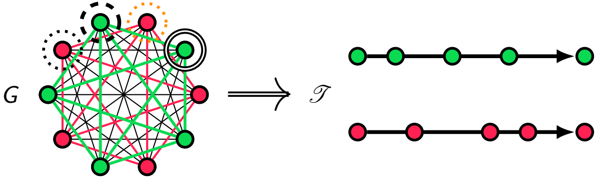

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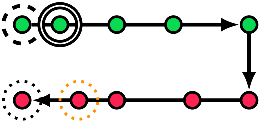

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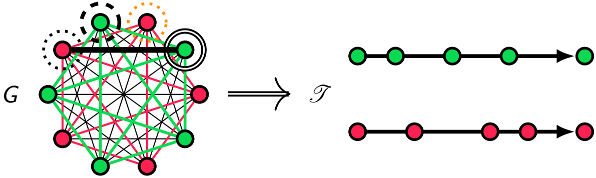

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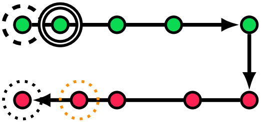

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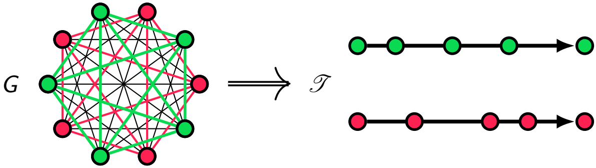

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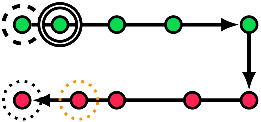

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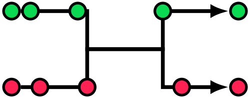

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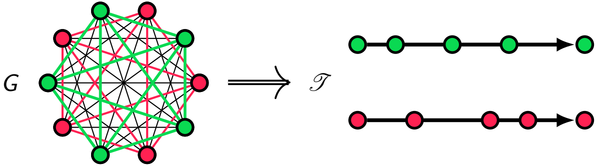

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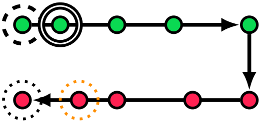

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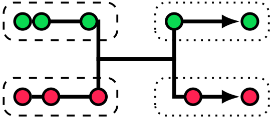

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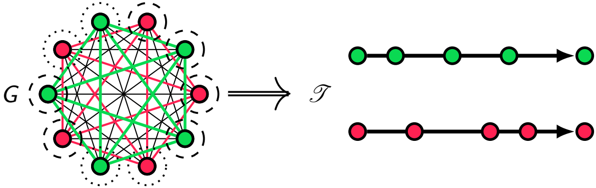

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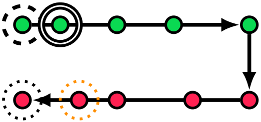

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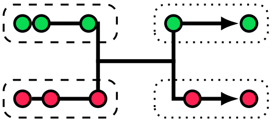

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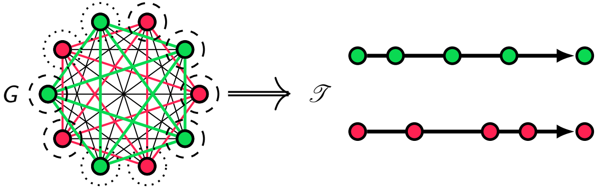

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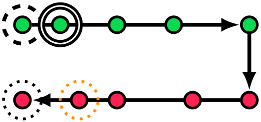

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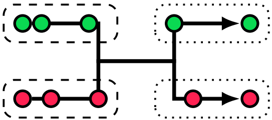

$\Rightarrow$ Each has to be complete bipartite in G, only possible if order ≤ 7, which we then manage.

Path Cover of temporal oriented trees (4) Conclusion

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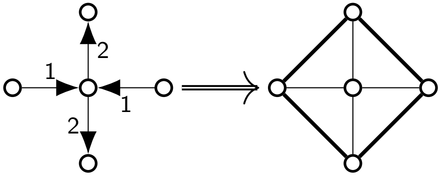
Remark

The connectivity graph is not chordal (it may contain C_4).



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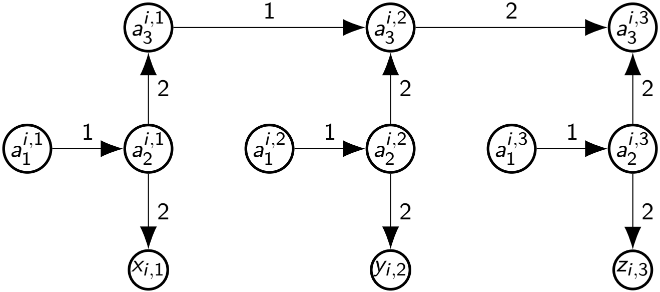

Figure: The gadget $H(s_i)$ for each triplet s_i.

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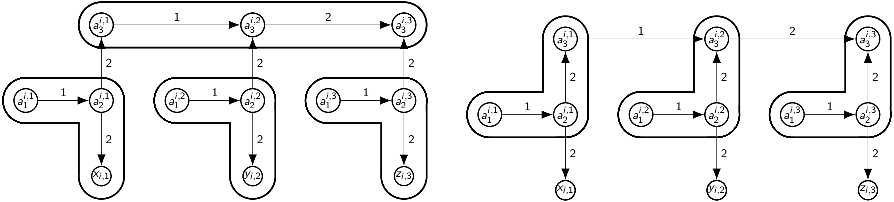


(a) $s_i \in M$ (b) $s_i \notin M$

Figure: Vertex partition of the gadget $H(s_i)$ into length-2 paths.

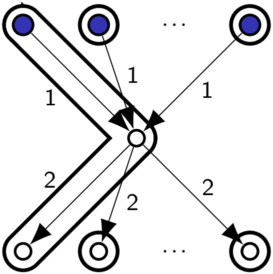
Total number of timesteps $= 2$ (bounded); Treewidth is unbounded; There are Temporal DAGs (transitive tournaments) without the Dilworth Property.

Hardness results (2)

Theorem [CDFK, 2024+]

Temporal Disjoint Path Cover is NP-hard on temporal oriented trees.

- Reduction from Unary Bin Packing (inspired by [Kunz, Molter, Zehavi, 23]);
- Treewidth = 1; Total number of timesteps = unbounded;
- Does not have TD-Dilworth property.



Parameterized complexity for TEMPORAL DISJOINT PATH COVER

Theorem [CDFK, 2024+]

TD-PC is FPT w.r.t. tw and $t_{\max}$ (total number of time-steps)

Dynamic programing on a nice tree decomposition

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Observation

Any arc of $\mathscr{D}$ appears in at most $t_{\max}$ paths of a TD-PC $\Rightarrow$ At most $p = \binom{\text{tw}}{2} \cdot t_{\max}$ temporally disjoint paths contain at least one arc from a given bag

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For simplicity, duplicate the arcs such that each has only one time label (so a TD-PC uses arc-disjoint paths)

Parameterized complexity for TD-PC (2) Type

Type: necessary information at each node v

$\Rightarrow$ At most types for any node

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Parameterized complexity for TD-PC (3) Consistency

Consistency of a type

- The ordered vertices V_i, the arcs of Q_i, and the information about the arcs going outside of X_v, induce temporal paths
- The arcs going outside of X_v exist in the digraph and their labels are compatible with the order
- Every vertex of X_v is in a V_i

Now, we compute from the bottom-up, maintaining consistency.

Dynamic programing using consistent types of partial solutions

- **Leaf node:** No partial solution since empty
- **Introduce node:** Check compatibility with the child (either *a* is in a path in the type, or *a* is added as a single-vertex path)
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Running time $2^{\mathcal{O}(p \log p)} n$, so FPT w.r.t. $p = f(\mathrm{tw}, t_{\max})$

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And for TPC?

Same principle, but the paths can intersect $\Rightarrow$ More information in type: how many times in the solution does Q_i appear $\Rightarrow$ Running time $k^{\mathcal{O}(p \log p)} n$ where $k \in \mathcal{O}(n)$ is the solution size $\Rightarrow$ XP w.r.t. p

Conclusion and future work

Temporal class	TPC	TD-PC
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Rooted trees	$\mathcal{O}(\ell n^2)$	$\mathcal{O}(\ell n^2)$
Oriented trees	$\mathcal{O}(\ell n^2 + n^3)$	NP-hard
DAGs*	NP-hard	NP-hard
Digraphs	XP (tw and $t_{\max}$) $n^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))}$	FPT (tw and $t_{\max}$) $2^{\mathcal{O}(\mathrm{tw}^2\, t_{\max} \log(\mathrm{tw}\, t_{\max}))}\, n$

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- Approximation?
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