Finding Geometric Representations of Apex Graphs is NP-Hard

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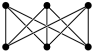

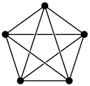

WALCOM 2022

25 March, 2022

Geometric intersection representation of a graph

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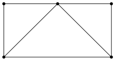
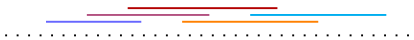
	Intervals on the real line	

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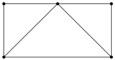
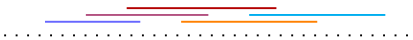
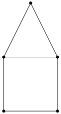
	Intervals on the real line	
	Intervals on the real line	×

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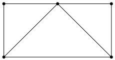
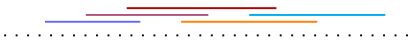
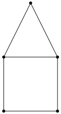
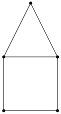
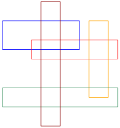
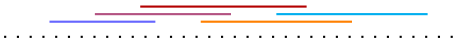
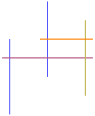
	Intervals on the real line	
	Intervals on the real line	×
	Rectangles on the plane	

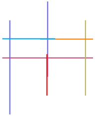
Examples



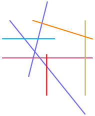
Interval graphs



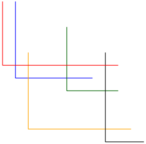
Pure-2-Dir



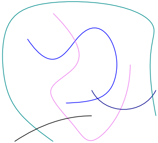
2-Dir



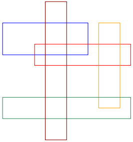
Segment graphs



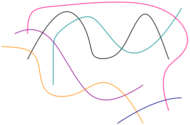
L-shapes



1-String



Rectangle intersection graphs



String graphs

Various representations of planar graphs

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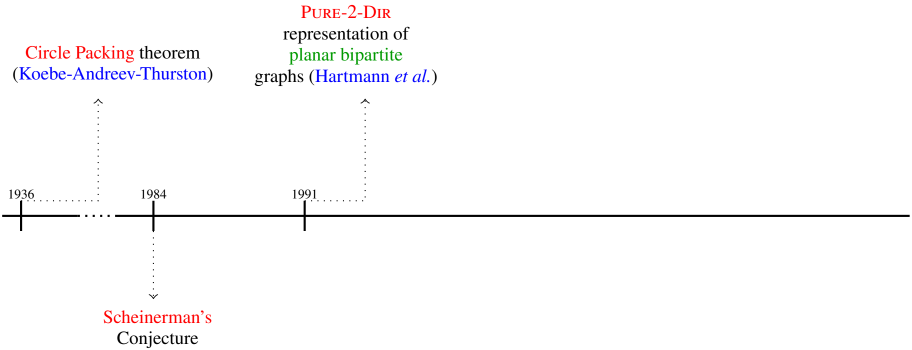

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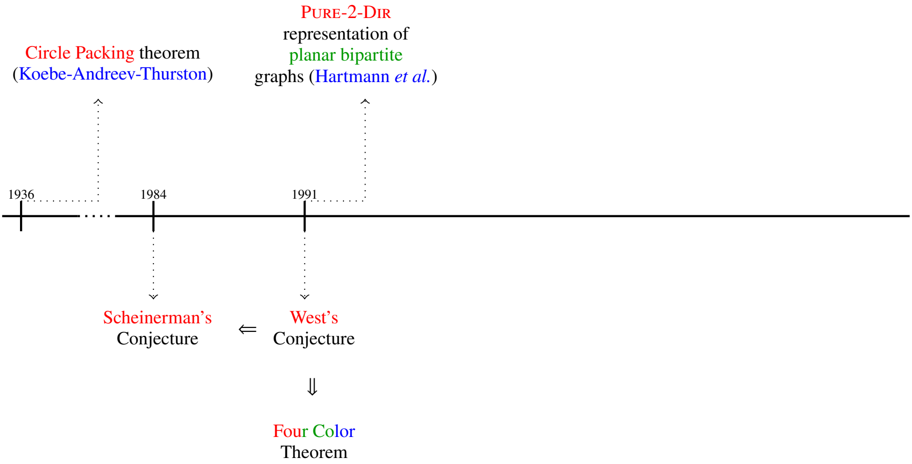

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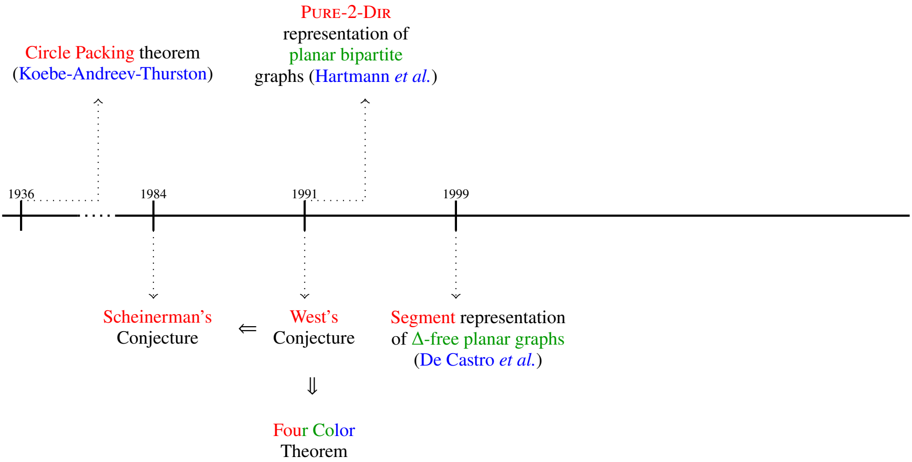

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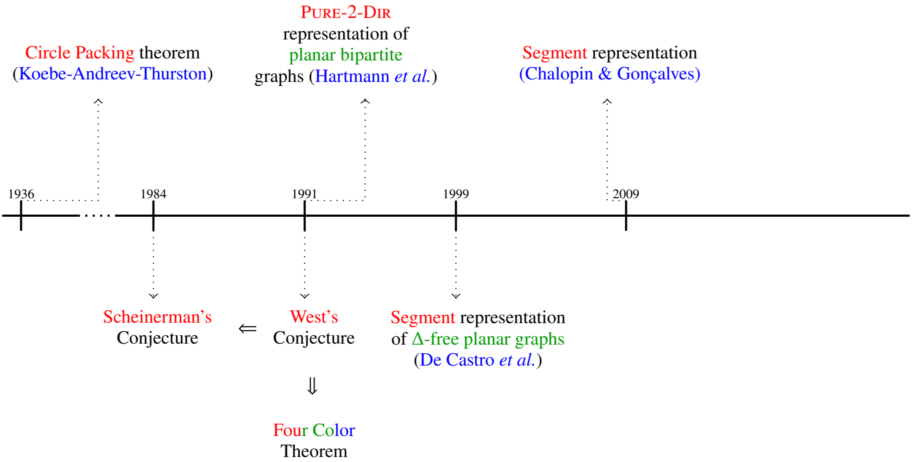

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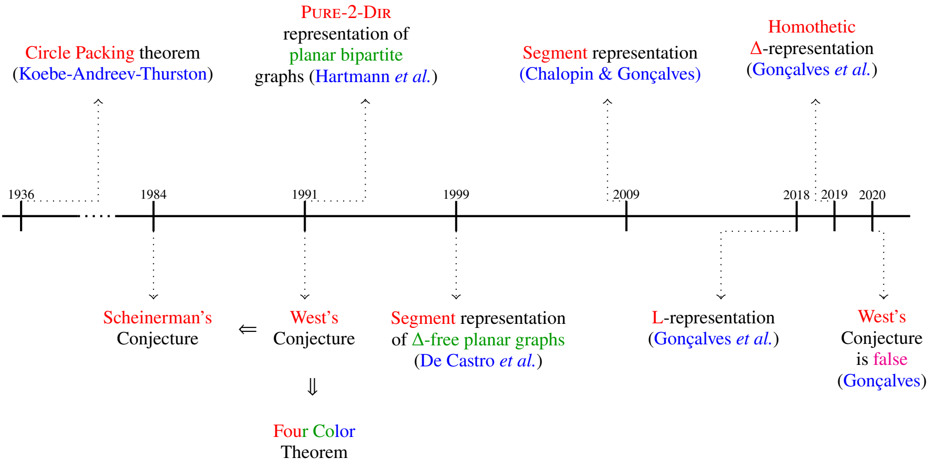

Can we generalize some of the theorems to apex graphs?

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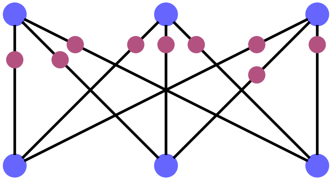
No !!

Observation

There are apex graphs that are not even string graphs.



We ask

Question

What are the computational complexities of representing apex graphs with various geometric objects?

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Formally,

Question

Given a geometric intersection graph class $\mathcal{G}$, what is the computational complexity of recognizing $\mathcal{G}$, when the inputs are restricted to apex graphs?

We answer (partially)

Theorem 1 (Main Result)

Let $\mathcal{G}$ be a graph class such that

$$\text{PURE-2-DIR} \subseteq \mathcal{G} \subseteq \text{1-STRING}.$$

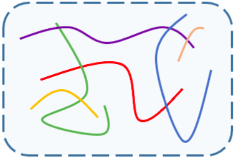
Then it is NP-hard to decide whether an input graph belongs to $\mathcal{G}$, even when the inputs are restricted to graphs that are both bipartite and apex.

PURE-2-DIR

$\subseteq$

$\mathcal{G}$

$\subseteq$



1-STRING

Recognition is NP-hard
even when input is bipartite and apex

Some corollaries

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Corollary

Recognizing intersection graphs of line segments is NP-hard, even when the inputs are restricted to graphs that are bipartite and apex.

Strengthening of a result by Kratochvíl and Matoušek (1989).

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Recognizing intersection graphs of L-shapes is NP-hard, even when the inputs are restricted to graphs that are bipartite and apex.

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Recognizing intersection graphs of L-shapes is NP-hard, even when the inputs are restricted to graphs that are bipartite and apex.

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Recognizing rectangle intersection graphs is NP-hard when the inputs are restricted to graphs that are bipartite and apex.

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Proof technique

Given a graph class $\mathcal{G}$ with PURE-2-DIR $\subseteq \mathcal{G} \subseteq$ 1-STRING.

- Reduce the PLANAR HAMILTONIAN PATH COMPLETION (PHPC) problem.

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Definition

PHPC is the following decision problem.
Input: A planar graph G.
Output: **Yes**, if G is a subgraph of a planar graph with a Hamiltonian path; **no**, otherwise.

Theorem (Auer & Gleißner, 2011)

PHPC is NP-hard.

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Objective

Given a planar graph G, construct a bipartite apex graph G_{apex} such that G is an yes-instance of PHPC if and only if $G_{apex} \in \mathcal{G}$.

Proof technique

We shall show the following:

Theorem 2

Given a planar graph G, we can construct a bipartite apex graph G_{apex} in polynomial time satisfying the following properties.

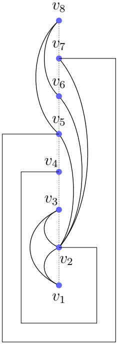
(a) If G_{apex} is in 1-STRING, then G is an yes-instance of PHPC.

(b) If G is an yes-instance of PHPC, then G_{apex} is in PURE-2-DIR.

$\Rightarrow$ Theorem 2 achieves the objective.

Proof: reduction

A planar graph $G \longrightarrow G_{3-div} \longrightarrow G_{apex}$



Planar graph G

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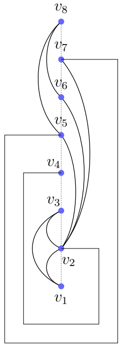

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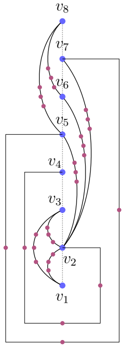

G_{3-div}

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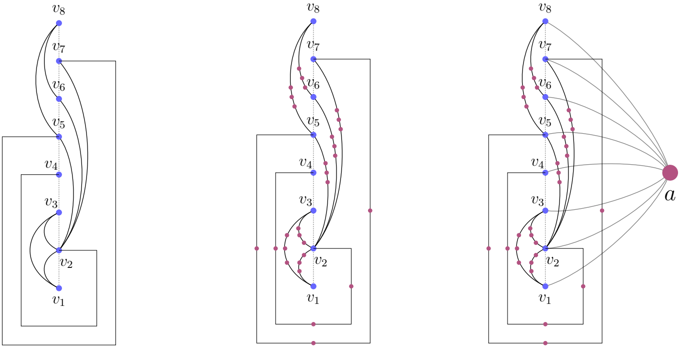

Planar graph G G_{3-div} G_{apex}

Proof: Theorem 2(a)

Objective

Show that if G_{apex} *is in 1-STRING, then* G *is an yes-instance of PHPC.*

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Objective

Show that if G_{apex} is in 1-STRING, then G is an yes-instance of PHPC.

Given a 1-STRING representation of G_{apex}, we will construct a planar super graph of G that contains a hamiltonian path.

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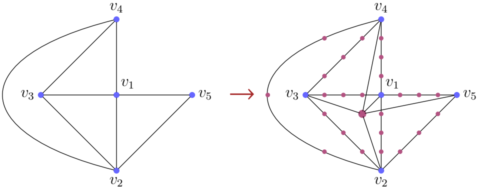

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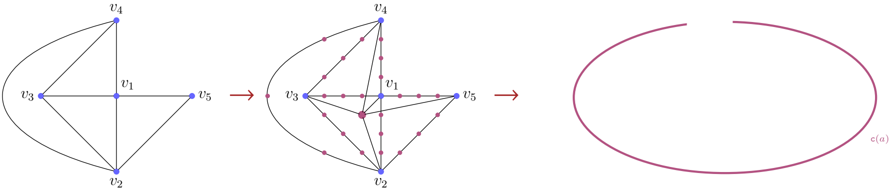

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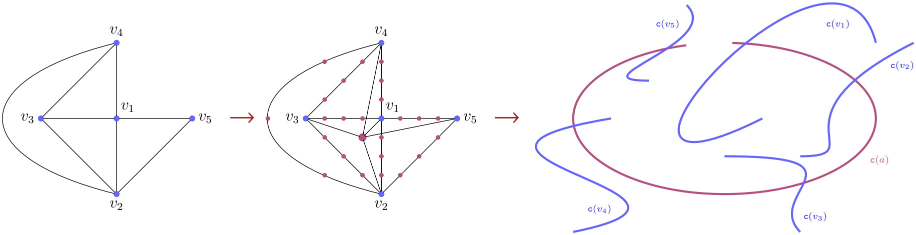

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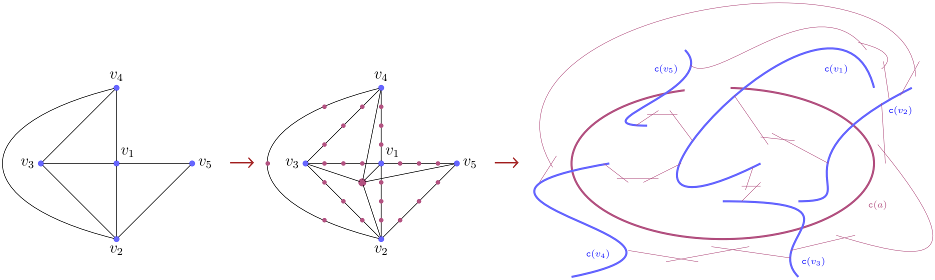

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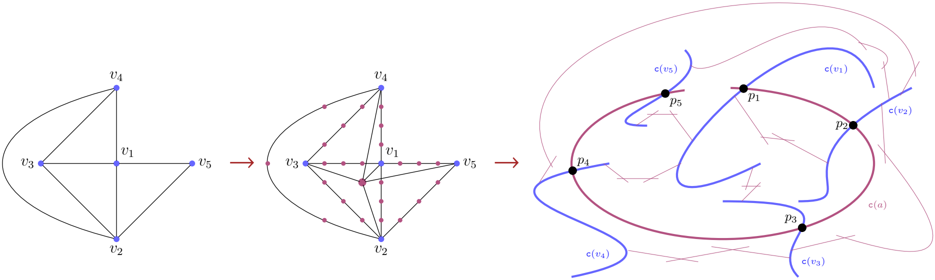

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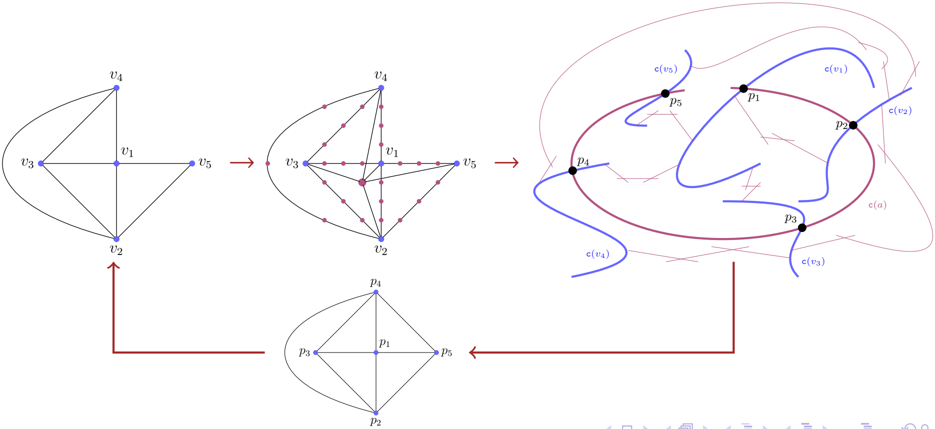

Proof: Theorem 2(b)

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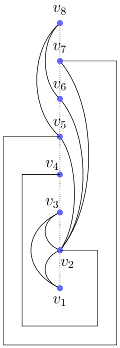

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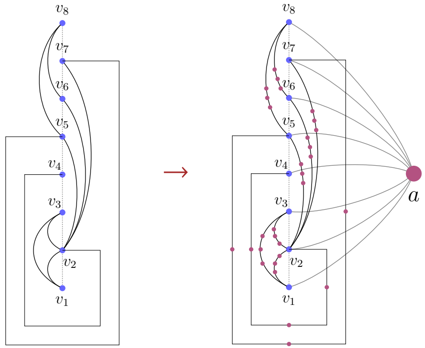

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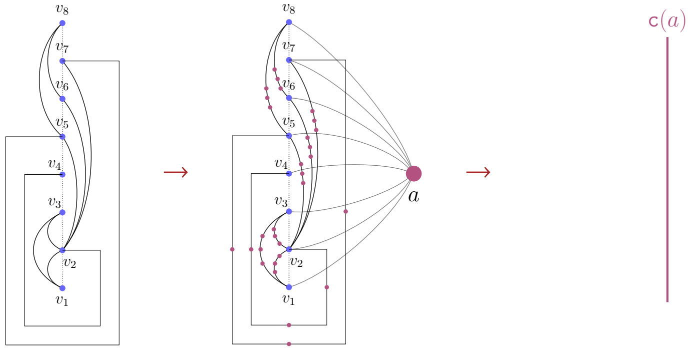

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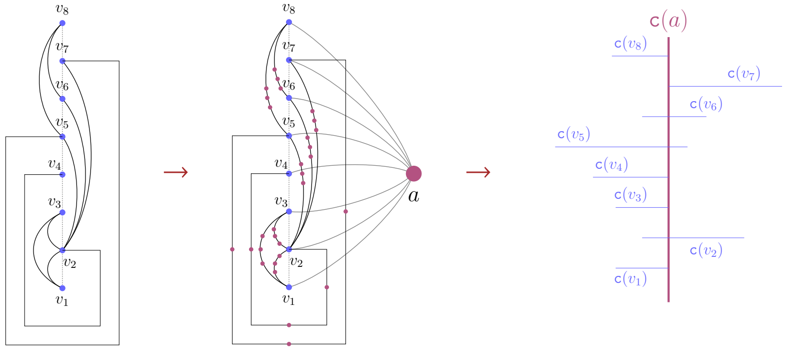

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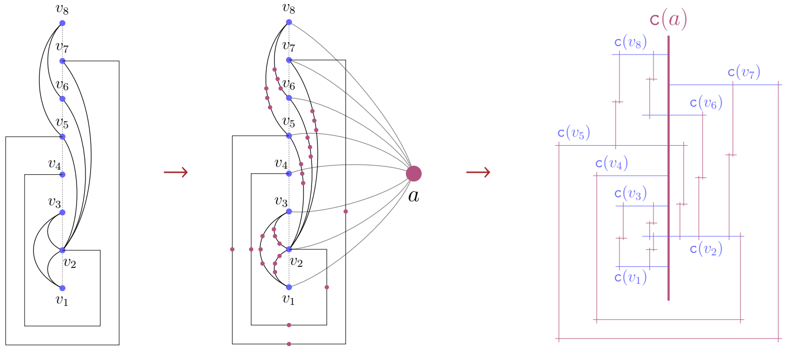

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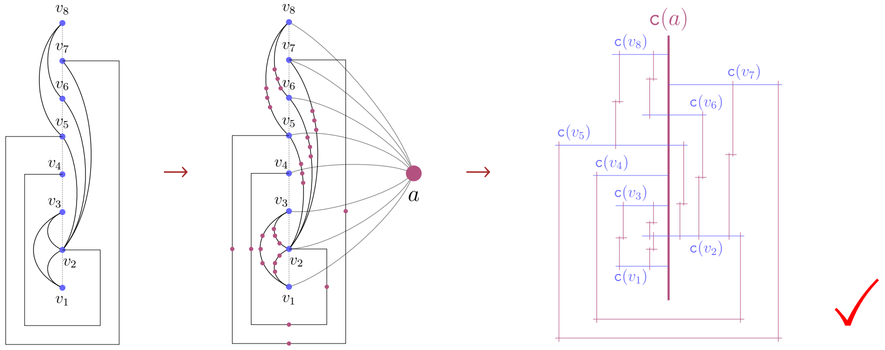

Open problems

1. Given a graph class $\mathcal{G}$ with PURE-2-DIR $\subseteq \mathcal{G} \subseteq$ STRING, what is the complexity of recognition of $\mathcal{G}$ when inputs are restricted to "almost planar graphs", e.g.
   - 1-planar graphs,
   - graphs with crossing number 1,
   - K_5-minor free graphs,
   - toroidal graphs,
   - projective planar graphs,

   $\vdots$
2. What is the complexity of recognizing intersection graphs of non-piercing regions when inputs are restricted to "almost planar graphs" ?

Thank you

Special thanks to

GRAPHMASTERS, 2020